A. S. Sanz

Investigating puzzling aspects of the quantum theory by means of its hydrodynamic formulation

Abstract Bohmian mechanics, a hydrodynamic formulation of the quantum theory, constitutes a useful tool to understand the role of the phase as the mechanism responsible for the dynamical evolution displayed by quantum systems. This role is analyzed and discussed here in the context of quantum interference, considering to this end two well-known scenarios, namely Young's two-slit experiment and Wheeler's delayed choice experiment. A numerical implementation of the first scenario is used to show how interference in a coherent superposition of two counter-propagating wave packets can be seen and explained in terms of an effective model consisting of a single wave packet scattered off an attractive hard wall. The outcomes from this model are then applied to the analysis of Wheeler's delayed choice experiment, also recreated by means of a reliable realistic simulation. Both examples illustrate quite well how the Bohmian formulation helps to explain in a natural way (and therefore to demystify) aspects of the quantum theory typically regarded as paradoxical. In other words, they show that a proper understanding of quantum phase dynamics immediately removes any trace of unnecessary artificial wave-particle arguments.

Keywords Bohmian mechanics · quantum phase · velocity field · interference · Young two-slit experiment · Wheeler delayed-choice experiment

1 Introduction

Quantum phenomena occur in real time. Although this may seem a trivial statement, monitoring the evolution in time of quantum systems in the laboratory has not been a feasible task until recently. The development and

Instituto de Física Fundamental (IFF-CSIC)
Serrano 123, 28006 Madrid, SPAIN
Tel.: +34-91-561-6800, ext. 941113
Fax: +34-91-585-4894
E-mail: asanz@iff.csic.es

improvement of highly refined experimental techniques have allowed us to explore time domains of the order of the femto- and attoseconds, at which the dynamics of many quantum processes and phenomena of interest take place (e.g., change of molecular configurations, charge transfer, entanglement dynamics, electron ionization by very intense laser fields, diffusion of adsorbed particles on surfaces, etc.). However, to obtain a full picture of quantum systems, it is necessary to perform a large number of measurements over hypothesized identical realizations of the same experiment or, equivalently, over many identical systems (of course, by "identical" it should be understood "nearly identical" in both cases). Statistics is the only way to extract relevant (i.e., physically meaningful) information from quantum systems, something that also happens when dealing with classical ensembles. It is a full collection of statistical data what constitutes the outcome compatible with the solutions provided by Schrödinger's equation. Therefore, any discussion about how every single event from within such a data collection evolves in time turns out to be nonphysical and, perhaps, even meaningless. But, is this totally true?

Young's two-slit experiment constitutes an ideal candidate to tackle the above question. In a real laboratory performance of this experiment, a beam of identical particles is launched against the two slits, observing far away behind them the appearance of the well-known interference fringes —an alternating pattern of regions of maximum and minimum density of recorded events (detected particles). According to Dirac [1], each one of these particles displays a wave-like nature, passing through *both* slits at the same time and interfering with themselves (*self-interference*) behind them. This model explains the observation of interference fringes. In addition, from von Neumann's collapse hypothesis [2] it follows that, at the detector, the particle wave function collapses at some random location. That is, the particle exhibits its corpuscular nature and behaves as a localized "piece" of matter. Though odd, nowadays Dirac's reasoning (plus the collapse postulate) constitutes the most widespread conception of how quantum systems behave. This oddity becomes even more striking after a closer look at the experiment, where the solutions described by quantum mechanics are built up particle by particle (event by event), keeping no coherence in time between two consecutive particles, even though they all come from the same source [3]. That is, particles are totally uncorrelated and, therefore, it is not possible to explain quantum interference by appealing to former physical interactions among them (entanglement) at the source. Experiments with photons [4–6], electrons [7, 8], ultracold atoms [9], or even with large molecular systems [10, 11] have all shown the universality of quantum interference as a phenomenon emergent from statistics, regardless of the size and complexity of the system investigated.

Such kind of experiments invites in a natural way to formulate and investigate descriptions of quantum phenomena in terms of statistical single-event realizations, implementing realistic numerical simulations of the experiment in order to gain some insight on the physical mechanics underlying the quantum phenomena investigated. In this regard, Bohmian mechanics, a hydrodynamic formulation of the quantum theory [12, 13], constitutes a reliable

and useful tool, where the evolution of quantum systems is represented in terms of streamlines. From a dynamical viewpoint, this formulation gives more relevance to the the quantum phase (and hence the quantum current density) than to the probability density. This pragmatic and natural use of Bohmian mechanics is analogous to the use of characteristics in other fields of physics and chemistry as an analytical tool [14], having nothing to do with the common view that Bohmian trajectories constitute some kind of "hidden" variables [15–17]. Bearing this in mind, here I analyze and discuss the role of the quantum phase as a mechanism involved in the dynamics displayed by quantum systems in the context of interference phenomena. Accordingly, it is observed that this phenomenon is analogous to dealing with effective barriers in Young-type experiments, in compliance with recently reported data on this experiment [18]. This study is subsequently used to analyze Wheeler's delayed choice experiment [19] in terms of a realistic numerical simulation, which removes any trace of paradox and explains in simple terms what happens inside the interferometer.

The remainder of this work has been organized as follows. The essentials of Bohmian mechanics and its contextualization with respect to the quantum theory are introduced and discussed in Section 2. In Section 3, the role of the quantum phase in relation to interference phenomena is discussed, introducing a new physical understanding of the notion of (quantum) superposition as well as the concept of effective dynamical potential (not to be confused with Bohm's usual quantum potential). In Section 4, a numerical simulation of Wheeler's delayed choice is analyzed taking into account the discussion of the previous section. Finally, some concluding remarks are summarized in Section 5.

2 Bohmian mechanics

Quantum mechanics admits different formulations. Each one emphasizes a way to conceive the quantum system and its evolution in time, although they all are equivalent —something similar can also be found in classical mechanics. For instance, while Schrödinger's wave mechanics allows to visualize the time-evolution of quantum systems, Heisenberg's matrix formulation provides us a point of view closer to that of classical mechanics, since the role of the classical variables is taken by the quantum operators. Dirac's formulation establishes a bridge between both and is of interest when dealing with open quantum systems, although Feynman's path representation is more powerful computationally. To establish a direct connection between quantum and classical systems (quantum-classical correspondence), we additionally have phase-space representations, such as the Wigner-Moyal or the Husimi ones.

In the particular case of Bohmian mechanics, what we have is a hydrodynamic description of quantum systems, where the system probability density is understood as a kind of fluid that spreads throughout the corresponding configuration space. Accordingly, there is an associated advective flux, namely the probability current density, which is a manifestation of a given velocity field acting on the probability density. This can easily be seen by

substituting the wave function in polar form,

$$\Psi(\mathbf{r},t) = \rho^{1/2}(\mathbf{r},t)e^{iS(\mathbf{r},t)/\hbar}, \tag{1}$$

into Schrödinger's equation (let us consider here only the nonrelativistic scenario for simplicity),

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\Psi. \tag{2}$$

This gives rise to two coupled, real-valued differential equations. One of them is the usual continuity equation,

$$\frac{\partial\rho}{\partial t} + \nabla\mathbf{J} = 0, \tag{3}$$

where $\mathbf{J} = \rho\nabla S/m$ is the probability current density playing the role of the aforementioned advective flux, associated with the velocity field $\mathbf{v} = \nabla S/m$. The other equation can be considered to be a quantum version of the Hamilton-Jacobi equation,

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} - \frac{\hbar^2}{2m}\frac{\nabla^2\rho^{1/2}}{\rho^{1/2}} + V = 0, \tag{4}$$

where the third term is Bohm's quantum potential. This equation led Bohm to postulate the existence of trajectories that could be identified with the actual particle motion, constituting a set of underlaying *hidden variables* that would then explain the causal evolution of the quantum system, although they would not be accessible to the experimenter. Based on Eq. (3), however, there is no necessity to establish such identification; the existence of a current $\mathbf{J}$ by itself allows us to define streamlines to analyze the evolution of the quantum system, as we also do when dealing with classical fluids or, in general, transport phenomena. These streamlines or trajectories are obtained after integrating the equation of motion

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\nabla S}{m} = \frac{\mathbf{J}}{\rho} = \frac{\hbar}{2im}\nabla\ln\left(\frac{\Psi}{\Psi^*}\right). \tag{5}$$

The main goal of the Bohmian formulation consists in dealing with quantum systems as if they were a kind of fluid given their delocalization in the corresponding configuration space. Swarms of streamlines or trajectories provide us with statistical information on how such a fluid evolves, indicating which regions of the configuration space are more highly populated or avoided at each time (i.e., where the probability density is higher or lower, respectively). Rather than true paths pursued by the system, such trajectories should be identified with the paths followed by some ideal tracer particles that move with the associated quantum flow [20], thus providing information about the latter —in the same sense that a leaf on a river tells us about the dynamics of the water flow, but does not reveal any information about the motion of the individual water molecules that constitute it. Nonetheless,

the strength of this representation relies on its closeness to classical statistical treatments, where physically meaningful quantities arise from ensembles rather than from single trajectories. Now, although physically irrelevant, such single trajectories are useful to infer properties associated with the system or process under study (in chemical reactivity, for instance, these trajectories allow to ascertain whether certain initial conditions lead to the formation of products or not). This is precisely the kind of information that can be expected from Bohmian trajectories, which is typically "hidden" within other formulations of quantum mechanics, although not incompatible with them at all. Of course, this idea transcends Bohmian mechanics; in the literature, it has been used with analogous purposes in different areas of physics and chemistry [14]. It is also in this sense that it would not be appropriate to consider Bohmian trajectories as hidden variables, because we can find exactly the same description in other fields.

3 Quantum interference, phase dynamics and the Bohmian non-crossing rule

Young's two-slit experiment is commonly explained appealing to the superposition principle: the waves diffracted by each slit superimpose and, depending on their phase at a given point, they may give rise to intensity maxima (equal phase at that point) or minima (different phase). This phenomenon is illustrated in Fig. 1 by means of a numerical simulation, where only the time-evolution along the transversal coordinate (parallel to the plane where the slits are allocated) has been considered. Specifically, the initial wave function, accounting for the diffraction at the slits, is described by a coherent superposition of two Gaussian wave packets [13],

$$\Psi(x,t) \propto e^{-(x-x_0)^2/4\sigma_0\tilde{\sigma}_t} + e^{-(x+x_0)^2/4\sigma_0\tilde{\sigma}_t}, \tag{6}$$

where σ_0 is the initial width of the wave packets (of the order of the slit width) and $\sigma_t = |\tilde{\sigma}_t|$ the width at a time t, with $\tilde{\sigma}_t = \sigma_0[1 + i(\hbar t/2m\sigma_0)]$. Because the expression (6) for the wave function is fully analytical, the field quantities represented in the panels of Fig. 1 can be readily determined by substituting into the corresponding (analytical) expressions the values of x and t where we want to evaluate them. As for the Bohmian trajectories, they have been numerically computed by means of a simple 4th-order Runge-Kutta scheme, which takes advantage of the analyticity of (6) to evaluate the right-hand side of the guidance equation (5) at each time step.

The development of interference fringes as a function of time is shown in the contour-plot of the probability density associated with (6), $\rho = |\Psi|^2$, displayed in Fig. 1a. The Bohmian trajectories superimposed in the figure (black solid lines) provide an accurate description of how such a probability density evolves from two localized regions to separate fringes that cover a larger area. Notice that the probability density is not simply an abstract concept, but has a very precise physical meaning: it tells how many events are registered within a certain region at a given time. This is in compliance with the fact that, in real life, detectors have a finite width and, therefore, at each

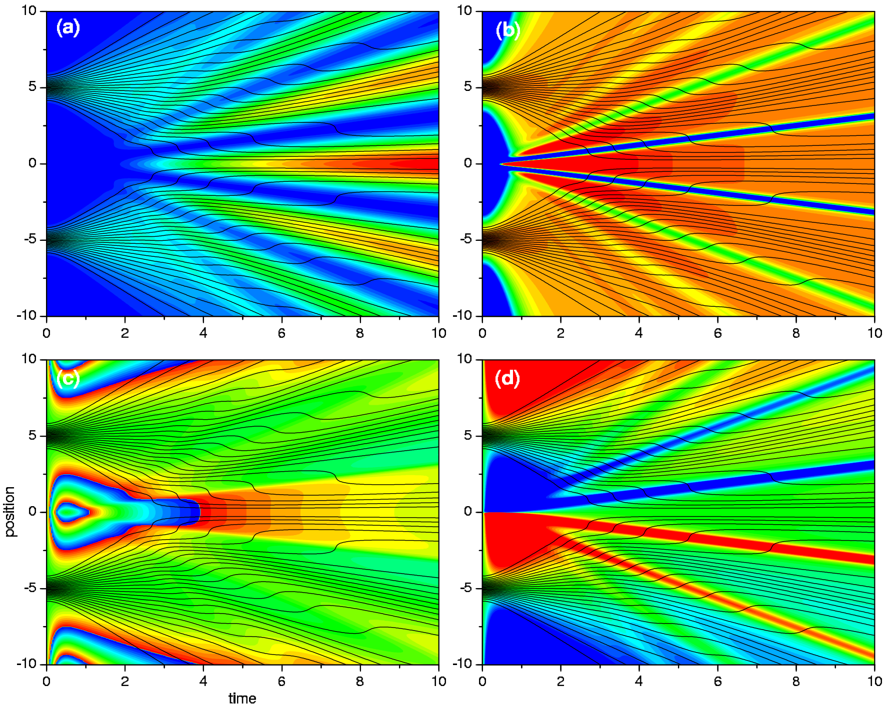

Fig. 1 Contour-plots of the probability density (a), quantum potential (b), quantum phase (c), and velocity field associated (d) with a coherent superposition of two Gaussian wave packets simulating Young's two-slit experiment. Sets of Bohmian trajectories leaving each slit are superimposed to provide a more vivid insight of the flow dynamics. In this simulation, the initial width of the wave packets is $\sigma_0 = 0.5$ and their centroids are at $|x_0| = 5$ (arbitrary units are considered without loss of generality, with $\hbar = m = 1$).

position they collect (during a fixed time) a number of events proportional to ρ. Numerical simulations aimed at providing a realistic description of diffraction by different types of systems [21, 22] show that, effectively, histograms built up with ensembles of Bohmian trajectories reproduce the theoretical predictions obtained from ρ.

Typically, the mechanical explanation for the particular evolution displayed by the trajectories relies on Bohm's quantum potential (see Fig. 1b). This potential is considered to be the mechanism leading the trajectories to eventually distribute along a series of plateaus and, therefore, to observe maxima (densely populated regions of nearly free motion), and minima (void regions between adjacent plateaus, where quantum forces are very intense) [22]. To some extent, this information is redundant, since the trajectories are streamlines connected to the current density, and hence their topology, will always be in agreement with how the latter evolves (i.e., in principle there is no need for appealing to a quantum potential). What is not that trivial here, however, is the physical role of the quantum phase, S, and its implica-

tions. As seen in Fig. 1c, independently of the value of ρ, S is well defined everywhere since the very beginning. The meaning of coherent superposition is linked to this fact: two waves are coherent if there is a continuity of phase, which makes impossible to consider both waves as independent entities. From this point of view, the longstanding debate about the role of the observer in Young's experiment is totally meaningless: the observer changes completely the experiment, breaking down such continuity of phase, and therefore making impossible the detection of eventual interference features.

Because of the non-additivity of S, there are two clearly distinguishable dynamical regions, as seen in Fig. 1d by means of the associated velocity field, $v = \dot{x}$, specified by Eq. (5). This naturally leads to conceive a single-slit model, where the flow leaving one of the slits is reflected back by an effective potential function that has nothing to do with the usual Bohm's potential. Specific details of this model can be found in [23]. Here it is enough to say that it consists of a square attractive well followed by an impenetrable wall located at $x = 0$. The well depth ($\mathcal{D}$) and width ($\mathcal{W}$) not only depend on time, but also on different initial physical parameters as

$$\mathcal{D}(t) = \frac{2\hbar^2}{m} \frac{1}{\mathcal{W}(t)}, \qquad \mathcal{W}(t) = \frac{\pi\sigma_t^2}{\dfrac{2|p_0|\sigma_0^2}{\hbar} + \dfrac{\hbar t}{2m\sigma_0^2}\,|x_0|}. \tag{7}$$

The second of these expressions has been obtained from a generalization of the coherent superposition (6), where initially both wave packets move towards $x = 0$ at the same speed (the corresponding initial momenta have the same absolute value, $|p_0|$, and opposite directions [23]). A plot of these two quantities for different values of $|p_0|$ is displayed in Fig. 2. According to this simple scattering model, the initial coherence induces an effect analogous to having two separate dynamical regions that can be independently associated. More importantly, the Bohmian trajectories coming from the upper slit cannot cross the point $x = 0$, and vice versa. This allows us to state that, even if we know nothing about the true individual motion of the quantum particles, at least at the level of the wave function, fluxes do not mix. This is precisely what Kocsis *et al.* [18] observed experimentally. Although it is impossible to accurately determine the true path pursued by a quantum particle, the fact that there is a continuity of the average transverse momentum in space at a given time physically means that quantum dynamics cannot be naively analyzed in terms of the superposition principle.

The experimental results reported in [18] are in compliance with the above model and the Bohmian formulation (although they are far away from constituting a confirmation of the real existence of Bohmian trajectories as the true paths followed by quantum particles). Testing the direct equivalence between a two-slit experiment and the scattering of a quantum system off an impenetrable attractive barrier at a quantum level, as described above, perhaps involves a lot of technical difficulties. A feasible, worth pursuing substitute in this regard could be an experiment in the line of those performed by Couder and Fort at the CNRS (France) and Bush at the MIT (USA) with bouncing droplets [24–30].

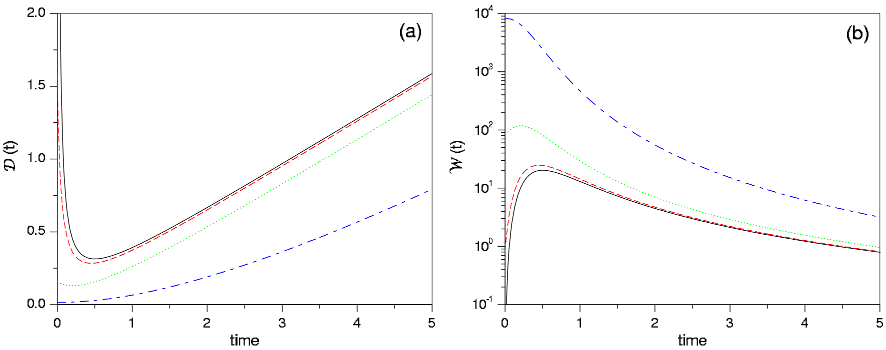

Fig. 2 Time-dependence of the width (a) and well depth (b) of the effective interference dynamical potential associated with the coherent superposition of Gaussian wave packets displayed in Fig. 1 when both wave packets move initially towards each other. Each curve refers to a different value of the momentum associated with the centroids of the wave packets: solid black: $|p_0| = 0$; dashed red: $|p_0| = 1$; dotted green: $|p_0| = 10$; dash-dotted blue: $|p_0| = 100$. Other parameters are as in Fig. 1.

4 Wheeler's delayed choice experiment revisited

The previous results are very useful now to understand and explain in a natural way Wheeler's delayed-choice experiment [19], removing any trace of paradoxical behavior. With this thought-experiment Wheeler wanted to reformulate one of the major issues of the Bohr-Einstein debates [31]: when does the quantum system make the choice to behave as a wave or as a particle? To this end, Wheeler conceived a clever experiment involving an optical Mach-Zehnder interferometer with a movable second beam splitter, and with a very dimmed light beam, so that at each time there is one and only one photon passing through the device. To understand the essence of the experiment and where the paradox arises, let us focus on the traditional schematics of the two interferometer configurations, displayed in Fig. 3, where the possible photon pathways are indicated in terms of optical (geometric) rays (this is a typical representation). Consider first that a photon enters the interferometer in the open configuration, illustrated in Fig. 3a. The beam splitter BS1, oriented at 45° with respect to the photon incidence direction, may produce either direct transmission towards a mirror M1, along a path P1 (denoted by the blue line), or a perpendicular deflection (reflection) towards a mirror M2, along P2 (red), with the same probability of 50%. In both cases, when the photon reaches the mirror (either M1 or M2), it undergoes a deflection of 90° with respect to the corresponding incidence direction. Eventually, the photon arrival will be registered with the same probability (50%) either by a detector D1, along P1, or a detector D2, along P2. This is a typical scenario where the photon would exhibit its corpuscular nature all the way through. Next, the experiment is slightly modified, inserting a second beam splitter, BS2, at the place where P1 and P2 intersect, as shown in Fig. 3b. Moreover, to avoid any phase-difference, the path length along P1 and P2 are the same between BS1 and BS2. From a classical viewpoint, the lack of phase-difference produces

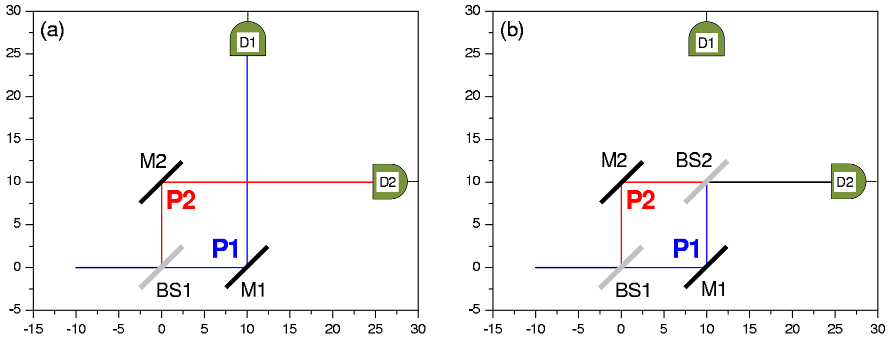

Fig. 3 Traditional optical (geometric) ray representation of the two scenarios considered by Wheeler in his delayed choice experiment [37]. The photon can be either transmitted or reflected by the first beam splitter (BS1) with the same probability of 50%. This generates two possible paths, P1 (blue) and P2 (red). (a) Open configuration: with absence of a second beam splitter, the probability to detect the photon at D1 or D2 is the same (50%). (b) Closed configuration: if a second beam splitter BS2 is introduced, the photon behaves as a wave, which interferes constructively along P2 and destructively along P1. In this case, all the detections are registered at D2.

that all (classical) light would reach D2. This result should be the same when the light beam is so weak that the experiment is reproduced photon by photon, which means that photons will be detected by D2. To explain this result, it is necessary to assume that the photon behaves as a wave. Accordingly, the beam splitter BS2 separates the horizontal and vertical wave components of the photon (just as BS1 did previously), which may come from P1 or from P2. After some simple algebra and the geometry of the setup, it is easy to see that the vertical components associated with the paths P1 and P2 are out-of-phase (180°) and cancel out, while the horizontal components are in-phase and their addition results in constructive interference, which explains why all photons are detected by D2. These two scenarios allow us to observe the dual nature of photons as well as, in general, any quantum particle. The "mystery" posed by Wheeler comes when BS2 is introduced or removed once the photon is already inside the interferometer. Wheeler's answer to this situation is that, regardless of when BS2 is put into play, the photon always behaves as it should, just like if it could somehow anticipate what is going to happen in future (inserting or removing BS2) and then behaving accordingly. That is, the photon makes a "delayed" choice, "taking its decision" on which aspect it will display, corpuscle or wave, only when BS2 has been removed or inserted, respectively. Nowadays this experiment is not a thought-experiment anymore; the puzzling and challenging dual behavior of quantum particles has been confirmed in the laboratory in many different ways [32–34].

The paradoxical behavior introduced by Wheeler readily dissipates taking into account the phase dynamics discussed in the previous section. From a conceptual, Bohmian point of view, this experiment was firstly discussed by Bohm and coworkers in 1985 [35], and then later on by Hiley and Callaghan [36]. According to the Bohmian non-crossing rule [23], appealed by this au-

thors, there is no paradox at all. When BS2 is absent, because the trajectories coming from P1 and P2 cannot cross the symmetry line at 45%, those coming from P1 are reflected in the direction of D2, and those from P2 in the direction of D1. That is, it is not that the photon follows P1 or P2 until it reaches the corresponding detector, as it is usually argued to introduce the corpuscular aspect, but there is an exchange in the directionality of the associated quantum flows, typical of the collision of two coherent wave packets [23], as discussed in Section 3. On the other hand, when BS2 is introduced, even in the case that the photon is already inside the interferometer, the recombination process of the two waves that takes place around this beam splitter produces that the two sets of trajectories will eventually go into only one of the detectors. This all-the-way wave behavior (notice that the traditional notion of corpuscle just disappears) is illustrated in Fig. 4 by means of realistic numerical simulations of the two processes described in the preceding paragraph [37]. Specifically, in this case, compared to the problem described in Section 3, the non-analyticity of the problem has led to consider more robust calculations employing the split-operator technique on a fixed grid to compute the evolution of the wave function. From this wave function, at each time step, the corresponding Bohmian trajectories (denoted by black solid lines in both panels of Fig. 4) were synthesized on-the-fly by means of a 4th-order Runge-Kutta method that was fed with interpolated values taken from neighboring grid points (this method has been proven to be quite stable and reliable in different atom-surface scattering problems [13]). As it can be inferred from these simulations, the photon does not make any choice at all. What happens is that there is a modification of the boundary conditions affecting its wave function, which simply gives rise to different outcomes, regardless of whether BS2 is introduced or removed once the wave function has started its evolution inside the interferometer. This kind of realistic simulations are very important to better understand the physics that is taking place in apparently paradoxical situations, as it has also been recently shown, for example, in the case of atomic Mach-Zehnder interferometry [38], used to discuss fundamental questions on complementarity [39, 40].

5 Concluding remarks

> "[...] we shall tackle immediately the basic element of the mysterious behavior in its most strange form. We choose to examine a phenomenon which is impossible, *absolutely* impossible, to explain in any classical way, and which has in it the heart of quantum mechanics. In reality, it contains the *only* mystery. We cannot make the mystery go away by "explaining" how it works. We will just tell you how it works. In telling you how it works we will have told you about the basic peculiarities of all quantum mechanics."

These sentences start chapter 2 of the third volume of Feynman's Lectures on Physics [41]. Effectively, the two-slit experiment probably constitutes the most elegant manifestation of the quantum nature of material particles. According to the traditional explanation of this experiment, what happens is

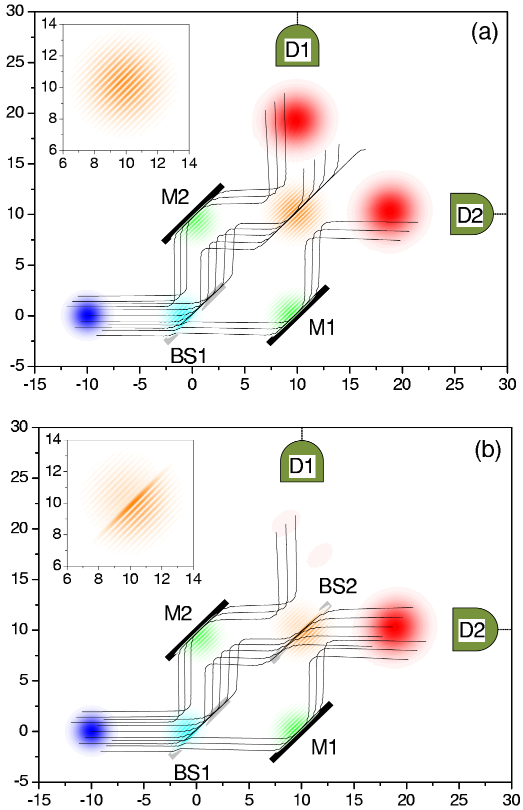

Fig. 4 Numerical simulation of the two scenarios considered by Wheeler in his delayed choice experiment [37]: (a) open configuration and (b) closed configuration. The background monochrome contour-plots correspond to different stages of the wave-function evolution inside the interferometer: blue: initial state (Gaussian wave packet); light blue: splitting at BS1; green: reflection at the mirrors (M1 and M2); orange: superposition of the two wave packets at the position where BS2 should be allocated; red: final stage (wave packets in their way to the corresponding detectors, D1 and D2). The insets show a magnification of the probability density in the region around BS2 in each case. The black solid lines denote ensembles of Bohmian trajectories starting with initial conditions covering different regions of the initial probability density (distribution).

that the particle, at some point before reaching the slits, behaves as a wave. The two outgoing diffracted waves then recombine again, giving rise to the typical interference fringes. This notion of single-particle *self-interference* is what Feynman had in mind when those above sentences were written, just the same as many other of the founders of quantum mechanics before, starting from Dirac, who stated that, in a beam of light consisting of a large number of photons, each photon only interferes with itself and not with the others [1]. This notion has prevailed until today, but is there still room for thinking quantum phenomena in a different way?

The different representations of the quantum theory provide us with different aspects of this theory, something similar to what we already know from the different classical approaches. Bohmian mechanics constitutes one of these representations, which stresses the role of the quantum phase, helping to understand how the system evolves throughout the corresponding configuration space and how the different elements (boundaries) influence its evolution. In particular, we have focused on quantum interference, showing how it emerges in Young's two-slit experiment and, based on the results observed in this renowned experiment, we have also analyzed Wheeler's delayed-choice experiment. By analyzing the topology of the corresponding trajectories, it is found that the phenomenon of quantum interference is analogous to dealing with effective barriers, helping to provide mechanical explanations and to remove paradoxical aspects of the quantum theory. It is worth stressing that the same "recipe" can be (has been) transferred to other fields of physics and chemistry with similar purposes [14], which leaves little room to keep thinking Bohmian mechanics as a hidden-variable theory.

Acknowledgements The author acknowledges support from the Ministerio de Economía y Competitividad (Spain) under Project No. FIS2011-29596-C02-01 as well as a "Ramón y Cajal" Research Fellowship with Ref. RYC-2010-05768.

References

1. Dirac, P.A.M.: The Principles of Quantum Mechanics, 4th ed., p. 9, Cambridge University Press, Cambridge (1958)
2. von Neumann, J.: Mathematische Grundlagen der Quantenmechanik. Springer Verlag, Berlin (1932). English translation by Beyer, R.T.: Mathematical Foundations of Quantum Mechanics. Princeton University Press, Princeton, NJ (1955)
3. Matteucci, G., Pezzi, M., Pozzi, G., Alberghi, G.L., Giorgi, F., Gabrielli, A., Cesari, N.S., Villa, M., Zoccoli, A., Frabboni, S., Gazzadi, G.C.: Build-up of interference patterns with single electrons. Eur. J. Phys. 34, 511–517 (2013)
4. Parker, S.: A single-photon double-slit interference experiment. Am. J. Phys. 39, 420–424 (1971)
5. Rueckner, W., Titcomb, P.: A lecture demonstration of single photon interference. Am. J. Phys. 64, 184–188 (1996)
6. Dimitrova, T.L., Weis, A.: The wave-particle duality of light: A demonstration experiment. Am. J. Phys. 76, 137–142 (2008)
7. Merli, P.G., Missiroli, G.F., Pozzi, G.: On the statistical aspect of electron interference phenomena. Am. J. Phys. 44, 306–307 (1976)
8. Tonomura, A., Endo, J., Matsuda, T., Kawasaki, T., Ezawa, H.: Demonstration of single-electron buildup of an interference pattern. Am. J. Phys. 57, 117–120 (1989). An original movies on the experiment can be found in the link: `https://www.youtube.com/watch?v=PanqoHa_B6c`
9. Shimizu, F., Shimizu, K., Takuma, H.: Double-slit interference with ultracold metastable neon atoms. Phys. Rev. A 46, R17–R20 (1992)
10. Arndt, M., Nairz, O., Vos-Andreae, J., Keller, C., van der Zouw, G., Zeilinger, A.: Wave-particle duality of C_{60} molecules. Nature 401, 680–682 (1999)
11. Juffman, T., Milic, A., Müllneritsch, M., Asenbaum, P., Tsukernik, A., Tüxen, J., Mayor, M., Cheshnovsky, O., Arndt, M.: Real-time single-molecule imaging of quantum interference. Nature Nanotech. 7, 297–300 (2012). A movie on the experiment can be found at the link: `https://www.youtube.com/watch?v=vCiOMQIRU7I`
12. Sanz, A.S., Miret-Artés, S.: A Trajectory Description of Quantum Processes. I. Fundamentals. Springer, Berlin (2012)

13. Sanz, A.S., Miret-Artés, S.: A Trajectory Description of Quantum Processes. II. Applications. Springer, Berlin (2014)
14. Sanz, A.S.: Particles, waves and trajectories: 210 years after Young's experiment. J. Phys.: Conf. Ser. 504, 012028(1–14) (2014)
15. Bohm, D.: A suggested interpretation of the quantum theory in terms of "hidden" variables. I. Phys. Rev. 85, 166–179 (1952)
16. Bohm, D.: A suggested interpretation of the quantum theory in terms of "hidden" variables. II. Phys. Rev. 85, 180–193 (1952)
17. Bohm, D., Vigier, J.P.: Model of the causal interpretation of quantum theory in terms of a fluid with irregular fluctuations. Phys. Rev. 96, 208–216 (1954)
18. Kocsis, S., Braverman, B., Ravets, S., Stevens, M.J., Mirin, R.P., Shalm, L.K., Steinberg, A.M.: Observing the average trajectories of single photons in a two-slit interferometer. Science 332, 1170–1173 (2011)
19. Wheeler, J.A.: The past and the delayed-choice double-slit experiment. In: Marlow, A.R. (ed.) Mathematical Foundations of Quantum Theory, pp. 9–48. Academic Press, New York (1978)
20. Sanz, A.S., Miret-Artés, S.: Quantum phase analysis with quantum trajectories: A step towards the creation of a Bohmian thinking. Am. J. Phys. 80, 525–533 (2012)
21. Sanz, A.S., Borondo, F., Miret-Artés, S.: Causal trajectories description of atom diffraction by surfaces. Phys. Rev. B 61, 7743–7751 (2000)
22. Sanz, A.S., Borondo, F., Miret-Artés, S.: Particle diffraction studied using quantum trajectories. J. Phys.: Condens. Matter 14, 6109–6145 (2002)
23. Sanz, A.S., Miret-Artés, S.: A trajectory-based understanding of quantum interference. J. Phys. A 41, 435303(1–23) (2008)
24. Couder, Y., Protière, S., Fort, E., Boudaoud, A.: Dynamical phenomena: Walking and orbiting droplets. Nature 437, 208– (2005)
25. Couder, Y., Fort, E.: Single-particle diffraction and interference at a macroscopic scale. Phys. Rev. Lett. 97, 154101(1–4) (2006)
26. Protière, S., Boudaoud, A., Couder, Y.: Particle-wave association on a fluid interface. J. Fluid. Mech. 554, 85–108 (2006)
27. Fort, E., Eddi, A., Boudaoud, A., Moukhtar, J., Couder, Y.: Path-memory induced quantization of classical orbits. Proc. Natl. Acad. Sci. USA 108, 17515–17520 (2010).
28. Bush, J. W. M.: Quantum mechanics writ large. Proc. Natl. Acad. Sci. USA 107, 17455–17456 (2010)
29. Harris, D. M., Moukhtar, J., Fort, E., Couder, Y., Bush, J. W. M.: Wave-like statistics from pilot-wave dynamics in a circular corral. Phys. Rev. E 88, 011011(R)(1–5) (2013)
30. Bush, J. W. M.: Pilot-wave hydrodynamics. Annu. Rev. Fluid Mech. 47, 269-292 (2015)
31. N. Bohr, Discussion with Einstein on epistemological problems in atomic physics, in *Albert Einstein: Philosopher-Scientist*, ed. P. A. Schilpp, Library of Living Philosophers, vol. 7 (MJF Books, New York, 1949), pp. 199-241. Reprinted in *Quantum Theory and Measurement*, eds. J. A. Wheeler and W. H. Zurek (Princeton University Press, Princeton, NJ, 1983), pp. 9-48 (see also commentaries on the issue in pp. 3-48).
32. Hellmuth, T., Walther, H., Zajonc, A., Schleich, W.: Delayed-choice experiments in quantum interference. Phys. Rev. A 35, 2532–2541 (1987)
33. Jacques, V., Wu, E., Grosshans, F., Treussart, F., Grangier, P., Aspect, A., Roch, J.-F.: Experimental realization of Wheeler's delayed-choice gedanken experiment. Science 315, 966–968 (2007)
34. Tang, J.-S., Li, Y.-L., Xu, X.-Y., Xiang, G.-Y., Li, C.-F., Guo, G.-C.: Realization of quantum Wheeler's delayed-choice experiment. Nature Photon. 6, 600–604 (2012)
35. Bohm, D., Dewdney, C., Hiley, B.J.: A quantum potential approach to the Wheeler delayed-choice experiment. Nature 315, 294–297 (1985)
36. Hiley, B.J., Callaghan, R.E.: Delayed-choice experiment and the Bohm approach. Phys. Scr. 74, 336–348 (2006)

37. A more detailed work on Wheeler's experiment, including a removable beam splitter, is currently being prepared.
38. Sanz, A.S., Davidović, M., Božić, M.: Full quantum mechanical analysis of atomic three-grating Mach-Zehnder interferometry. Ann. Phys. 353, 205–221 (2015)
39. Chapman, M.S., Hammond, T.D., Lenef, A., Schmiedmayer, J., Rubenstein, R.A., Smith, E., Pritchard, D.E.: Photon scattering from atoms in an atom interferometer: Coherence lost and regained. Phys. Rev. Lett. 75, 3783–3787 (1995)
40. Cronin, A.D., Schmiedmayer, J., Pritchard, D.E.: Optics and interferometry with atoms and molecules. Rev. Mod. Phys. 81, 1051–1129 (2009)
41. Feynman, R.P., Leighton, R.B., Sands, M.: The Feynman Lectures on Physics, vol. 3, ch. 2. Addison, New York (1960s) For the online version, see: http://www.feynmanlectures.caltech.edu/III_02.html